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Operation Separable Problems in the Nonlocal Calculus of Variations*

DOMINIC G. B. EDELEN

Center for the Application of Mathematics, Lehigh University, Bethlehem, Pennsylvania 18017

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1. INTRODUCTION AND SUMMARY OF PREVIOUS RESULTS

Basically, the nonlocal calculus of variations considers questions of stationarity of functionals and leads to Euler equations that are integro-differential in nature. An interesting and important subclass of nonlocal variational problems consists of those problems in which the differential structure and the integral structure can be separated: the Euler equations can be treated as differential equations whose solutions must satisfy a system of integral constraints. See references [1–3] for examples of such problems. Separability of the differential and integral operations leads to significant computational simplifications and affords a clarification of the properties of the solutions in terms of whether they arise from the differential or from the integral structure. The combined differential and integral structure, however, still leads to marked differences between the properties of the solutions of nonlocal problems and local ones. We shall give specific examples of systems with one independent variable in which the boundary value problem either has no solutions or more than one solution.

A quick summary of the essentials of the nonlocal calculus of variations is given below for the convenience of the reader. The details can be found in Refs [4–11].

Let D be an open, simply connected, n-dimensional region of n-dimensional number space E_n referred to a coordinate system (x), and let D^* denote the closure of D with respect to the Euclidean topology of E_n. The boundary of D^* is denoted by ∂D and is assumed to be a smooth $(n - 1)$-dimensional subspace of E_n with the exception of a finite number of edges. Let $\{\phi_A(x^m)\}$ $A = 1,..., N$ be an N–tuple of C^1 functions whose domains of

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definition contain D^*. It is convenient in what follows to use the notation $\{{}_\alpha\phi_A(x^m)\}$, where $\alpha = 0, 1, \ldots, n$ and

$$ {}_0\phi_A \equiv \phi_A\,, \; {}_m\phi_A \equiv \partial_m\phi_A \equiv \partial\phi_A/\partial x^m. \tag{1.1}$$

Let $g_a = g_a(x^m, z^m, {}_\alpha\phi_A(z^m))$, $a = 1, \ldots, q$ be q given functions of class C^2 in their $2n + N(n+1)$ arguments. These functions serve to define q functionals by the relations

$$\begin{aligned} k_a(x^m; \phi_A) &= \int_{D^*} g_a(x^m, z^m, {}_\alpha\phi_A(z^m))\, dV(z), \\ dV(z) &= dz^1\, dz^2 \cdots dz^n. \end{aligned} \tag{1.2}$$

Let $L = L(x^m, {}_\alpha\phi_A(x^m), k_a)$ be a given function of class C^2 in its $n + N(n+1) + q$ arguments. This function serves to define an "action" functional by the relation

$$J[\phi_A](L) = \int_{D^*} L(x^m, {}_\alpha\phi_A(x^m), k_a(x^m; \phi_A))\, dV(x), \tag{1.3}$$

where the notation $J[\phi_A](L)$ is used to emphasize the dependence of this functional on the choice of L. The occurrence of the k's as arguments of L gives rise to the nonlocal variational calculus, for we wish to stationarize the value of J relative to the choice of $\{\phi_A(x^m)\}$ from the subspace of C^1 functions that assume given values on D^*. The norm in use is the uniform convergence norm for these functions and their first partial derivatives. The set $S(L)$, comprised of all elements of the function space that render $J[\phi_A](L)$ stationary in value is referred to as the *stationarization set base L*.

A statement of the conditions for stationarization of J in a convenient form requires certain additional notation. We first define the functions $g_a{}^*$ by the relations

$$g_a{}^* = g_a(z^m, x^m, {}_\alpha\phi_A(x^m)), \qquad g_a = g_a(x^m, z^m, {}_\alpha\phi_A(z^m)). \tag{1.4}$$

Since the k's are, in general, functions of x^m's, the assumed functional form of L shows that $\partial L/\partial k_a$ is a function of x^m, ${}_\alpha\phi_A(x^m)$, and $k_a(x^m; \phi_A)$, and hence is a function of x^m for any given $\{\phi_A(x^m)\}$. The quantity

$$\frac{\partial L}{\partial k_a}(z^m)$$

is then obtained by replacing x^m by z^m in $\partial L/\partial k_a$:

$$\frac{\partial L}{\partial k_a}(z^m) = \frac{\partial}{\partial k_a} L(z^m, {}_\alpha\Phi_A(z^m), k_a(z^m; \Phi_A)).$$

The *local Euler–Lagrange operators* are defined by

$$\{e \mid W(v_r)\}_{\phi_A}(x^m) - \frac{\partial W}{\partial \phi_A(x)} - \frac{\partial}{\partial x^m}\Big(\frac{\partial W}{\partial(\partial_m \phi_A(x))}\Big), \tag{1.5}$$

where the v_r's are quantities that are to be held constant during the indicated differentiation with respect to $\{\phi_A\}$ and $\{\partial_m \phi_A\}$. The *nonlocal Euler–Lagrange operators* are then defined by

$$\{E \mid L\}_{\phi_A}(x^m) = \{e \mid L(k_a)\}_{\phi_A}(x^m) + \int_{D^*} \frac{\partial L}{\partial k_a}(z^m)\{e \mid g_a{}^*\}_{\phi_A}(x^m)\, dV(z). \tag{1.6}$$

The summation convention is assumed to apply to all indexed quantities, thus, the index m is to be summed from one through n in the right side of (1.5) and the index a is to be summed from one through q on the right side of (1.6). It can then be shown, under fairly weak continuity conditions, that if $\phi_A(x^m)$ are such that the *nonlocal Euler equations*,

$$\{E \mid L\}_{\phi_A}(x^m) = 0, \qquad A = 1,\dots, N, \tag{1.7}$$

are satisfied, then $\{\phi_A(x^m)\}$ belongs to $S(L)$.

2. Operation Separation of the Euler Equations

In the general case, the q functions g_a have both the x's and the z's as explicit arguments, and hence the integrals occurring in the Euler equations cannot be simplified. We now restrict our attention to the class of problems in which the functions g_a do not depend on the x's explicitly:

$$g_a = g_a(z^m, {}_\alpha\phi_A(z^m)). \tag{2.1}$$

For this class of problems, (1.4) gives us

$$g_a{}^* = g_a(x^m, {}_\alpha\phi_A(x^m)), \tag{2.2}$$

and hence $\{e \mid g_a{}^*\}_{\phi_A}(x^m)$ is independent of the z^m's for every N-tuple of functions $\{\phi_A(x^m)\}$. In this instance, the nonlocal Euler–Lagrange operators, (1.6), become

$$\{E \mid L\}_{\phi_A}(x^m) = \{e \mid L(k_a)\}_{\phi_A}(x^m) + \{e \mid g_a{}^*\}_{\phi_A}(x^m) \int_{D^*} \frac{\partial L}{\partial k_a}(z^m)\, dV(z). \tag{2.3}$$

Now, by (1.2) and (2.1), the q quantities k_a are constants for any given $\{\phi_A(x^m)\}$, and the same holds true for the q quantities.

$$K_a = \int_{D^*} \frac{\partial L}{\partial k_a}(z^m)\, dV(z). \tag{2.4}$$

The Euler equations thus assume the form

$$\{e \mid L_a(k)\}_{\phi_A}(x^m) + K_a\{e \mid g_a{}^*\}_{\phi_A}(x^m) = 0. \tag{2.5}$$

For any given $\{\phi_A(x^m)\}$, the k_a's and the K_a's are given numbers, and hence they are given numbers for any $\{\phi_A(x^m)\}$ that satisfy the Euler equations (2.5). We may thus consider the k_a's and the K_a's as constants in (2.5) provided that we append to the system (2.5) the relations that define the k_a's and the K_a's in terms of $\{\phi_A(x^m)\}$. If we solve the Euler equations (2.5) with the k_a's and the K_a's treated as constants (that is, we solve the system of differential equations (2.5)), we could obtain a system of functions

$$\phi_A = U_A(x^m, k_a, K_a), \tag{2.6}$$

since the values of the k_a's and the K_a's would enter into the solution of (2.5). The k_a's and the K_a's are not arbitrary numbers, however, for they must be those numbers that result from substituting (2.6) into the equations defining the k_a's and the K_a's, namely (1.2) and (2.4). We must thus adjoin the conditions

$$k_a = \int_{D^*} g_a(z^m, {}_\alpha U_A(z^m, k_a, K_a))\, dV(z). \tag{2.7}$$

$$K_a = \int_{D^*} \frac{\partial L}{\partial k_a}(z^m)\, dV(z). \tag{2.8}$$

What we have done, in effect, is to *separate* the *differential operators* involved in the Euler equations from the *integral operators* that are involved in these equations. It is clear that this separation is always possible whenever the functions g_a have the form given by (2.1). We therefore refer to nonlocal Euler equations as being *operation separable* whenever $g_a = g_a(z^m, {}_\alpha\phi_A(z^m))$, in which case the nonlocal Euler equations reduce to the system of differential equations

$$\{e \mid L(k_a)\}_{\phi_A}(x) + K_a\{e \mid g_a{}^*\}_{\phi_A}(x) = 0, \qquad A = 1,\ldots, N \tag{2.9}$$

whose solutions must satisfy the system of integral constraints

$$k_a = \int_{D^*} g_a(z^m, {}_\alpha\phi_A(z^m))\, dV(z), \qquad a = 1,\ldots, q, \tag{2.10}$$

$$K_a = \int_{D^*} \frac{\partial L}{\partial k_a}(z^m)\, dV(z), \qquad a = 1,\ldots, q. \tag{2.11}$$

From the standpoint of the nonlocal Euler equations as systems of integro-differential equations, operation separation allows significant simplification since we can solve the differential equations (2.9) without having to worry about the integral structure. The integral structure then appears as the system of constraints (2.10) and (2.11). The imposition of the system of constraints (2.10) and (2.11) changes the whole complexion of the class of problems relative to the similar class of problems based only on the Euler equations (2.9) (i.e., based on equations that can be obtained from the local calculus of variations with the Lagrangian function $L + K_a g_a{}^*$). In general, when initial or boundary data is adjoined to the system (2.9), the constraints (2.10) and (2.11) will preclude whole classes of boundary or initial values, and will further compound the issue by leading to nonuniqueness of the boundary value or the initial value problem. We shall illustrate these properties of nonlocal systems in the next Section.

3. Linear, Operation Separable Euler Equations

This Section examines the properties of the linear, operation separable Euler equations that result under the following prescription: $n = 1$, $N = 1$, $q = 2$, $\phi_1(x^1) = y$, $x^1 = x$,

$$g_1(z, {}_\alpha y(z)) = y(z), \; g_2(z, {}_\alpha y(z)) = y(z)^2, \tag{3.1}$$

$$k_1 = \int_0^T g_1 \, dx = \int_0^T y(z)\, dz, \qquad k_2 = \int_0^T y(z)^2 \, dz, \tag{3.2}$$

$$L = \tfrac{1}{2}(y')^2 - f_1(k_1, k_2) - f_2(k_1, k_2)\, y(x) - f_3(k_1, k_2)\, y(x)^2, \tag{3.3}$$

where y' denotes the derivative of $y(x)$ with respect to x. When these quantities are substituted into (1.6) (1.7), we obtain the Euler equation

$$y'' + \lambda y + \gamma = 0, \tag{3.4}$$

where λ and γ are functions of k_1 and k_2 that are given by

$$\begin{aligned} \lambda &= 2f_3 + T\frac{\partial f_1}{\partial k_2} + k_1 \frac{\partial f_2}{\partial k_2} + k_2 \frac{\partial f_3}{\partial k_2} \\ \gamma &= f_2 + T\frac{\partial f_1}{\partial k_1} + k_1 \frac{\partial f_2}{\partial k_1} + k_2 \frac{\partial f_3}{\partial k_1}. \end{aligned} \tag{3.5}$$

When (3.4) is considered in the operation separable fashion, it is then *linear*.

For $\lambda \neq 0$, (3.4) has the general solution

$$y(x) = -\frac{\gamma}{\lambda} + Ae^{x\sqrt{\lambda}} + Be^{-x\sqrt{\lambda}} \tag{3.6}$$

to which we have to adjoin the conditions

$$y(0) = a = -\frac{\gamma}{\lambda} + A + B, \tag{3.7}$$

$$y(T) = b = -\frac{\gamma}{\lambda} + Ae^{T\sqrt{\lambda}} + Be^{-T\sqrt{\lambda}}, \tag{3.8}$$

$$k_1 = -\frac{T\gamma}{\lambda} + \frac{A}{\sqrt{\lambda}}(e^{T\sqrt{\lambda}} - 1) - \frac{B}{\sqrt{\lambda}}(e^{-T\sqrt{\lambda}} - 1), \tag{3.9}$$

and

$$k_2 = \left(\frac{\gamma^2}{\lambda^2} + 2AB\right) T + \frac{A^2}{2\sqrt{\lambda}}(e^{2T\sqrt{\lambda}} - 1) - \frac{B^2}{2\sqrt{\lambda}}(e^{-2T\sqrt{\lambda}} - 1)$$
$$- \frac{2\gamma}{\lambda^{3/2}} A(e^{T\sqrt{\lambda}} - 1) + \frac{2\gamma}{\lambda^{3/2}} B(e^{-T\sqrt{\lambda}} - 1), \tag{3.10}$$

where λ and γ are given in terms of k_1 and k_2 by (3.5). Equations (3.9) and (3.10) are obtained from substitution of (3.6) into the Eqs. (3.2) that define k_1 and k_2. In this particularly simple situation, we obtain the system of four equations (3.7)–(3.10) for the determination of the unknowns A, B, k_1, and k_2 in terms of the initial and final values a and b. The intrinsic integral nonlinearity of this system is what gives rise to the interesting effects.

As a particular instance, consider the situation in which we have

$$f_1 = 0, f_2 = \rho(k_1)^2, f_3 = 0. \tag{3.11}$$

We then have $L = \frac{1}{2}(y')^2 - \rho y k_1^2$ so that the functional J is given by

$$J[y](L) = \int_0^T \{\tfrac{1}{2}(y'(x))^2 - \rho y(x)\, k_1^2\}\, dx$$
$$= \tfrac{1}{2}\int_0^T (y(x)')^2\, dx - \rho \int_0^T y(x)\, dx \left(\int_0^T y(z)\, dz\right)^2$$
$$= \tfrac{1}{2}\int_0^T (y')^2\, dx - \rho k_1^3. \tag{3.12}$$

Since k_2 does not occur explicitly, we can simply ignore all k_2 dependence in the above formula. The relations (3.5) give us $\lambda = 0, \gamma = 3\rho k_1^2$, and hence the operation separated Euler equation, (3.4) becomes

$$y'' = -3\rho k_1^2.$$

We thus obtain

$$y(x) = a + Bx - 3\rho k_1{}^2 x^2/2. \tag{3.13}$$

where we have used the condition $y(0) = a$. The remaining constraints (3.8) and (3.9) give

$$b = a + BT - 3\rho k_1{}^2 T^2/2 \tag{3.14}$$

$$k_1 = \int_0^T y(z)\, dx = aT + BT^2/2 - \rho k_1{}^2 T^3/2. \tag{3.15}$$

When (3.14) and (3.15) are solved for B and k_1, we obtain

$$k_1 = \frac{2}{\rho T^3}\left\{1 \pm \sqrt{1 - \rho T^4\left(\frac{a+b}{2}\right)}\right\}, \tag{3.16}$$

$$B = \frac{b-a}{T} + \frac{6}{\rho T^5}\left\{1 \pm \sqrt{1 - \rho T^4\left(\frac{a+b}{2}\right)}\right\}^2, \tag{3.17}$$

and hence

$$y(x) = a + \left[\frac{b-a}{T} + \frac{6}{\rho T^5}\left\{1 \pm \sqrt{1 - \rho T^4\left(\frac{a+b}{2}\right)}\right\}^2\right] x$$
$$- \frac{6}{\rho T^6}\left\{1 \pm \sqrt{1 - \rho T^4\left(\frac{a+b}{2}\right)}\right\}^2 x^2. \tag{3.18}$$

Accordingly, *a real solution exists only if*

$$\rho\left(\frac{a+b}{2}\right) \leqslant \frac{1}{T^4}, \tag{3.19}$$

in which case there are two solutions to the given boundary value problem. For ρ positive, we get the definite barrier (3.19) in the space of boundary data which depends on the *mean value* of the initial and final values. Further, when the boundary data satisfies (3.19), there are always two solutions. This situation is to be compared with the solution of the corresponding local problem whose Euler equation is $y'' = -3\rho k_1{}^2$, $y(0) = a$, $y(T) = b$, where k_1 is not to be identified with the integral of $y(x)$ over $(0, T)$. For the local problem, there exists a unique solution for any given pair of finite numbers (a, b).

What we have shown, in effect, is that the nonlocal calculus of variations leads to situations in which there are only restricted values of the boundary data for which solutions exist, and satisfaction of the Euler equation and the boundary data does not guarantee a unique solution. A wealth of new problems is thus presented by the nonlocal variational calculus in even the simplest of cases.

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